Eigenvectors in Bottleneck Algebra

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ABSTRACT

Let $(B, \leqslant)$ be a nonempty, linearly ordered set without maximum and minimum, and $(\oplus, \otimes) = (\max, \min)$. A vector x is said to be an eigenvector of a square matrix A if $A \otimes x = x$. The aim of the present paper is to characterize the eigenvectors by means of the associated graph of the matrix and to give bounds for the set of all eigenvectors. We define the lower and the upper basic eigenvectors and derive efficient methods for computing them.

1. INTRODUCTION

The classical linear-algebraic problem—for a given square matrix A find a number λ (called the *eigenvalue*) and a vector x (called the *eigenvector*) such that

$$Ax = \lambda x, \tag{1}$$

—has been studied for decades, and its importance in various branches of mathematics as well as in applications is well recognized. Since the beginning of the 1960s, several authors have shown independently that some practical questions can be formulated formally in the same way as the classical eigenvector-eigenvalue problem (*eigenproblem* for short) if we appropriately choose the operations of "addition" and "multiplication"; see e.g. [2].

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Although many results have been derived for the eigenproblem in nonclassical structures (see e.g. [3–6]), little is known about the eigenproblem in general, as noted in [8]. The most universal results seem to be those presented by Gondran and Minoux in [5]. They work in the structure $(S, \oplus, \otimes)$, the so-called *semiring*, where both operations $\oplus$ and $\otimes$ are associative, they possess neutral elements, and $\oplus$ is commutative and idempotent.

In [4] an interpretation of eigenvectors in the semiring of real numbers $(\mathbb{R} \cup \{-\infty\} \cup \{\infty\}, \max, \min)$ was described for the hierarchical classification of vectors, similar to that in factor analysis. We now recall the most important results concerning the eigenvectors in semirings, as derived in [5].

Let a square matrix A over S be given. Denote

$$A^{(k)} = A \oplus A^2 \oplus \cdots \oplus A^k,$$

and let the symbol A^+ stand for $\lim_{k \to \infty} A^{(k)}$, if it exists, and $A^* = E \oplus A^+$, where E is the unit matrix.

It was shown in [5] that if A^* exists, then

(a) the columns of A^* multiplied by some constant are eigenvectors of A, and

(b) every eigenvector of A can be expressed as a "linear" combination of columns of A^*.

However, the existence of A^* for every matrix A over S was not proved, and moreover, the question of efficient computing of eigenvectors remained untouched. The aim of the present paper is to characterize the eigenvectors in terms of the associated graph and to derive an alternative way of computing them. We also derive some upper and lower bounds for all eigenvectors of a given matrix.

2. DEFINITIONS AND NOTATION

The quadruple $\mathscr{B} = (B, \oplus, \otimes, \leqslant)$ is called a *bottleneck algebra* (BA for short) if $(B, \leqslant)$ is a nonempty, linearly ordered set without maximum and minimum and $\oplus$ and $\otimes$ are binary operations on B defined by the formulas

$$a \oplus b = \max\{a, b\},$$

$$a \otimes b = \min\{a, b\}.$$

(Note that this definition explicitly excludes the neutral elements from the structure.)

The term bottleneck algebra was introduced in [1] to emphasize the connection of this algebraic structure with classical problems of combinatorial optimization such as the bottleneck assignment problem, the bottleneck traveling salesman problem, or bottleneck location problems.

The most common cases of bottleneck algebras are the set $\mathbb{Z}$ of all integers and the set $\mathbb{R}$ of all real numbers with the natural order, and our examples will also use these structures.

If n is a positive integer, denote by N the set $\{1, 2, \dots, n\}$. Both operations $\otimes$ and $\oplus$ are extended to the set $B(n)$ of all n-tuples over $\mathscr{B}$ and the set $B(n, n)$ of all square matrices of order n as in conventional linear algebra. Our n-tuples are column vectors, and the notation x^{T} for $x \in B(n)$ will denote the transpose of x.

For a given matrix $A \in B(n, n)$, recall the notion of the *associated graph* $G(A)$: $G(A) = (V, H)$ is the strongly complete directed weighted graph with the node set $V = N$, the arc set $H = N \times N$, and capacity a_{ij} assigned to each arc (i, j).

For a given matrix $A \in B(n, n)$ and a value $h \in B$, define the *associated threshold graph* $G(A, h) = (V, H(A, h)$ as the directed graph on the node set $V = N$ where the arc set $H(A, h)$ is defined by the condition

$$(i, j) \in H(A, h) \quad \text{if and only if} \quad a_{ij} \geqslant h.$$

This definition implies immediately that $G(A, h)$ consists of n isolated nodes for every $h > \max\{a_{ij};\ i, j \in N\}$, and $H(A, h) = N \times N$ if $h \leqslant \min\{a_{ij};\ i, j \in N\}$. It can also be easily seen that a given matrix A determines at most $n^2 + 1$ different threshold graphs, since $G(A, h)$ does not change for h varying between two consecutive values of entries of A. Moreover, for $h' > h$, $G(A, h')$ is a subgraph of $G(A, h)$.

A sequence of nodes $p = (i_1, i_2, \dots, i_{m+1})$ of a graph $G = (V, H)$ is called a *path* if $(i_k, i_{k+1}) \in H$ for all $k = 1, 2, \dots, m$. These arcs, together with the nodes i_k for $k = 1, 2, \dots, m + 1$, are said to be *on the path* p. The *length* of a path, denoted by $l(p)$, is the number of arcs on it; in our case $l(p) = m$. If all nodes on a path p are pairwise distinct, then p is called *elementary*. It is clear that the length of an elementary path in a graph with n nodes is at most $n - 1$. A path p with $i_1 = i_{m+1}$ is called a *cycle*; we do not demand that all nodes on a cycle be distinct.

In a weighted graph G a *capacity* $c(p)$ can be assigned to each path p in the following way:

$$c(p) = \prod\nolimits^{\otimes} \{a_{ij};\ (i, j) \text{ on } p\}.$$

We say that a node $x \in V$ is *precyclic* in the graph G if there exists a path $(x = u_0, u_1, \dots, u_k, \dots, u_{k+s}, u_k)$ for some nodes $u_1, u_2, \dots, u_{k+s}$ in G. In other words, x is precyclic if a cycle can be reached from v, traversing a path. In this definition $k = 0$ is also allowed; hence if v lies on a cycle, then v is precyclic too.

3. BASIC PROPERTIES

Suppose that a matrix $A \in B(n, n)$ is given. It is clear that no entry of $A \otimes x$ can be greater than $\max\{a_{ij};\ i, j = 1, \dots, n\}$, so for eigenvalues $\lambda > \max\{a_{ij};\ i, j = 1, \dots, n\}$ and the corresponding eigenvectors x one always has $\lambda \otimes x = x$; hence these eigenvectors are simply the solutions of the system of equations

$$A \otimes x = x. \tag{2}$$

If a vector x is a solution of (2), then for every $\lambda \in B$ we have

$$A \otimes (\lambda \otimes x) = \lambda \otimes (A \otimes x) = \lambda \otimes x = \lambda \otimes (\lambda \otimes x),$$

because of the distributivity of $\otimes$ with respect to $\oplus$ and of the idempotency of $\otimes$. Hence, every $\lambda \in B$ is an eigenvalue of any matrix A for which (2) is solvable, and at least one eigenvector (although possibly not all) corresponding to λ can be found easily, once we know the solutions of (2).

For the sake of brevity, in what follows, we shall use the term "eigenvector of a matrix A" only for the solutions of (2).

Our first theorem shows that in a bottleneck algebra every matrix has infinitely many eigenvectors of a simple form, which can be determined easily.

THEOREM 1. *For every matrix $A \in B(n, n)$ denote $c(A) = \prod^{\otimes}_{i \in N} \sum^{\oplus}_{j \in N} a_{ij}$. Every constant vector $x = (\alpha, \alpha, \dots, \alpha)^{\mathrm{T}}$ with $\alpha \leqslant c(A)$ is an eigenvector of A, and no constant vector with entries $\alpha > c(A)$ is an eigenvector of A.*

Proof. Notice that $c(A)$ is the minimum row maximum in A, and take $x = (\alpha, \alpha, \dots, \alpha)^{\mathrm{T}}$ for some $\alpha \leqslant c(A)$. Since $(A \otimes x)_i = \sum^{\oplus}_{j \in N} (a_{ij} \otimes x_j)$, and in each row there is at least one index j with $a_{ij} \geqslant c(A)$, we have $A \otimes x = x$. On the other hand, if $\alpha > c(A)$, then there exists a row i with all entries not greater than $c(A)$. In this case, $(A \otimes x)_i \leqslant c(A) < \alpha$; hence x cannot be an eigenvector. ∎

EXAMPLE 1. Let

$$A = \begin{pmatrix} 3 & 1 & 8 \\ 2 & 5 & 7 \\ 4 & 3 & 5 \end{pmatrix} \qquad \text{over } \mathbb{Z}$$

be given. Its row maxima are 8, 7, 5; hence $c(A) = 5$. It can be easily verified that $A \otimes (5,5,5)^{\mathrm{T}} = (5,5,5)^{\mathrm{T}}$, while for $x = (\alpha, \alpha, \alpha)^{\mathrm{T}}$ when $\alpha > 5$ we have $(A \otimes x)_3 = 5 < \alpha$; hence x is not an eigenvector of A.

In what follows, we will call the constant eigenvectors from Theorem 1 *trivial eigenvectors* of A. The greatest of them will be called the *lower basic eigenvector* of A and will be denoted by $x_*(A)$.

We now proceed in our search for nontrivial eigenvectors of A.

4. THE UPPER BASIC EIGENVECTOR

DEFINITION 1. Let $A \in B(n,n)$ be given. $x^*(A)$ is a vector defined in the following way: for all $k \in N$

$$x_k^*(A) = \max\{h \in B;\ k \text{ is precyclic in } G(A,h)\}.$$

The argument "(A)" of the vector $x^*(A)$ will usually be omitted if no confusion can arise. Before deriving some properties of $x^*(A)$, we illustrate its definition by several examples.

EXAMPLE 2. Find the vector $x^*(A)$ for the matrix A from Example 1. If we write the entries of A in descending order (without repetitions), we obtain 8, 7, 5, 4, 3, 2, 1. Associated threshold graphs $G(A,h)$ for $h > 5$ are acyclic; for illustration $G(A,7)$ is given in Figure 1(a). $G(A,5)$ is displayed in Figure 1(b), and we see that now all its nodes are precyclic; hence $x^*(A) = (5,5,5,)^{\mathrm{T}}$. In this case we obtain $x^*(A) = x_*(A)$.

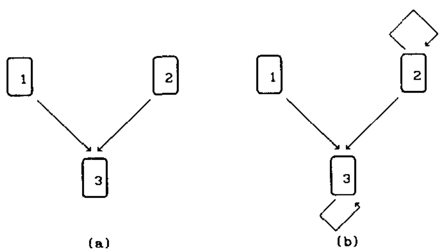

FIG. 1.

EXAMPLE 3. The situation for

$$A = \begin{pmatrix} 1 & 3 & 8 & 6 \\ 0 & 4 & 5 & 2 \\ 5 & 7 & 6 & 1 \\ 9 & 4 & 3 & 2 \end{pmatrix}$$

is somewhat more complicated. The ordered entries of A are 9, 8, 7, 6, 5, 4, 3, 2, 1, 0. The graphs $G(A,7)$, $G(A,6)$, and $G(A,5)$ are displayed in Figure 2, and we can see that for $h > 6$ the threshold graphs are acyclic. The three nodes 1, 3, and 4 are precyclic in $G(A,6)$; hence $x_1^*(A) = x_3^*(A) = x_4^*(A) = 6$. Node 2 is precyclic for the first time in $G(A,5)$; thus $x_2^*(A) = 5$. We obtain $x^*(A) = (6,5,6,6)^{\mathrm{T}}$.

THEOREM 2. *For every matrix $A \in B(n,n)$, the vector $x^*(A)$ is an eigenvector of A.*

Proof. Choose an arbitrary $k \in N$. According to the definition of x^*, there exists a path $p = (k, j_1, \dots, j_s, j_{s+1}, \dots, j_s)$ in $G(A, x_k^*)$, such that the sequence $j_s, j_{s+1}, \dots, j_s$ forms a cycle. For every node j_i on p, it holds that j_i is precyclic in the graph $G(A^*, x_k)$ too; hence $x_{j_i}^* \geqslant x_k^*$. Thus we have

$$(A \otimes x^*)_k = \sum_{j \in N}^{\oplus} \left(a_{kj} \otimes x_j^*\right) \geqslant a_{kj_1} \otimes x_{j_1}^* \geqslant x_k^*;$$

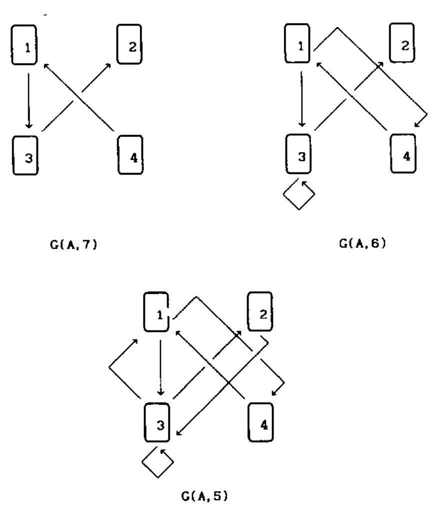

FIG. 2.

the last inequality holds because $a_{kj_1} \geqslant x_k^*$ [as the pair (k, j_1) is an arc in $G(A, x_k^*)$] and $x_{j_1}^* \geqslant x_k^*$. Now it remains only to show that $(A \otimes x^*)_k$ cannot be strictly greater than x_k^*.

On the contrary suppose that $(A \otimes x^*)_k = a_{kj} \otimes x_j^*$ for some j such that $a_{kj} > x_k^*$ and $x_j^* > x_k^*$. According to Definition 1 we have that j is precyclic in $G(A, x_j^*)$; let $p = (j = j_0, j_1, \dots, j_s, j_{s+1}, \dots, j_s)$ be the path ensuring this property. Recall that $a_{j_i j_{i+1}} \geqslant x_j^*$ for all (j_i, j_{i+1}) on p. After inserting the arc (k, j) at the beginning of p, we obtain a new path $p' = (k, j, \dots, j_s, \dots, j_s)$. The entries of A corresponding to the arcs on p' are all (except a_{kj}) greater than or equal to x_j^*; hence k is precyclic in $G(A, h)$, where $h = a_{kj} \otimes x_j^* > x_k^*$—a contradiction with the definition of the vector $x^*(A)$. ■

LEMMA 1. *Let $A \in B(n, n)$ and $x \in B(n)$. If $A \otimes x = x$, then for all $k \in N$ the node k is precyclic in $G(A, x_k)$.*

Proof. Since $x_k = (A \otimes x)_k = \sum_{j \in N}^{\oplus} (a_{kj} \otimes x_j)$, we have for at least one j that $x_k = a_{kj} \otimes x_j$. Denote one index for which this equation is fulfilled by j_1. Thus $x_k \leqslant a_{kj_1}$ and $x_k \leqslant x_{j_1}$; hence $(k, j_1) \in H(A, x_k)$. Similarly, for j_1

$$x_{j_1} = (A \otimes x)_{j_1} = \sum_{j \in N}^{\oplus} (a_{j_1 j} \otimes x_j).$$

Again the equation $x_{j_1} = a_{j_1 j} \otimes x_j$ is fulfilled by at least one index j; denote an arbitrary one of them by j_2. Now we have $x_{j_1} \leqslant a_{j_1 j_2}$, $x_{j_1} \leqslant x_{j_2}$, and with the previous inequality $x_k \leqslant x_{j_1}$ we obtain $a_{j_1 j_2} \geqslant x_k$; hence $(j_1, j_2) \in H(A, x_k)$. After repeating similar considerations several times, a sequence of nodes $k, j_1, j_2, \dots$ is generated. At the latest at the nth position of this sequence a repetition must occur, as we are choosing from the set of n nodes. Since for all m we have $a_{j_m j_{m+1}} \geqslant x_k$, we have found a cycle in $G(A, x_k)$ attainable from node k, so k is precyclic in $G(A, x_k)$. ■

THEOREM 3. *Let $A \in B(n, n)$, $x \in B(n)$. If $A \otimes x = x$ then $x \leqslant x^*(A)$.*

Proof. By Lemma 1, if $A \otimes x = x$, then every node k is precyclic in the graph $G(A, h)$ for $h = x_k$. Since $x_k^*(A)$ is equal to the greatest $h \in B$ with this property, we have $x_k \leqslant x_k^*(A)$ for all $k \in N$. ■

So we have proved that $x^*(A)$ gives an upper bound for all eigenvectors of A, and now we are entitled to call the vector $x^*(A)$ the *upper basic eigenvector* of the matrix A.

On the basis of what has been said so far, we can describe a method for computing the vector $x^*(A)$ for a given matrix A. After arranging all entries of A in descending order, a sequence, say $h_1, h_2, \dots, h_m$, is obtained, and now the threshold graphs $G(A, h_1), G(A, h_2), \dots$ can successively be constructed. In each graph $G(A, h_k)$ we find out whether there are some new nodes which have become precyclic. If so, we assign the value h_k to the respective entries of $x^*(A)$ and repeat the procedure for h_{k+1}. The process can be finished when $x^*(A)$ is fully defined.

In the worst case, this method has to search n^2 graphs with n nodes. Therefore its time complexity is not greater than $O(n^4)$, but the procedure is rather complicated for implementation. So we proceed to search for a simpler method.

5. ANOTHER METHOD FOR COMPUTING THE UPPER BASIC EIGENVECTOR

DEFINITION 2. Let $A \in B(n, n)$ be given. The *A-generated sequence* is a sequence of vectors $x^{(1)}(A)$, $x^{(2)}(A), \dots$ defined as follows:

$$x_i^{(1)}(A) = \sum_{j=1}^{n}{}^{\oplus} a_{ij} \qquad \text{for all} \quad i \in N,$$

and further by induction

$$x^{(k+1)}(A) = A \otimes x^{(k)}(A) \qquad \text{for all} \quad k = 1, 2, \dots .$$

Again, the argument "(A)" will be omitted if it is understood from the context.

The A-generated sequence will be helpful for computing the vector $x^*(A)$, so we start with the study of its properties. Let us first find an interpretation of the vectors $x^{(k)}(A)$ in the graph $G(A)$.

Denote by $P_j^{(k)}$ the set of all paths of length k (not necessarily elementary) beginning at node j. Clearly, $x_j^{(1)}(A)$ is the greatest capacity of a path in $P_j^{(1)}$. Suppose now that $x_j^{(k)}(A)$ is the greatest capacity of a path in $P_j^{(k)}$, for all $j \in N$. If we compute

$$(A \otimes x^{(k)})_i = \sum_{j \in N}{}^{\oplus} \left(a_{ij} \otimes x_j^{(k)}\right),$$

then the term $a_{ij} \otimes x_j^{(k)}$ can be interpreted in this way: we take the arc (i, j) and connect it at the beginning of the path $p \in P_j^{(k)}$ which has the greatest capacity of all paths from $P_j^{(k)}$. It is clear that the length of the obtained path is $k + 1$; it begins at node i and has the greatest capacity of all paths with length $k + 1$ beginning with the arc (i, j). If we now take the maximum over all $j \in N$, we obtain the greatest capacity of a path in $P_i^{(k+1)}$.

Obviously, $x_i^{(k)} \leqslant x_i^{(k-1)}$ for all $i \in N$ and k, since each path p in $P_i^{(k)}$ [with capacity $c(p)$] can be split into a path p' from $P_i^{(k-1)}$ [with capacity $c(p')$] and one arc e [with weight $c(e)$] at its other end; but adding an arc cannot raise the capacity of a path, since $c(p) = c(p') \otimes c(e) \leqslant c(p')$. Hence we have

$$x_i^{(1)} \geqslant x_i^{(2)} \geqslant \cdots \geqslant x_i^{(n)} \geqslant x_i^{(n+1)} \geqslant \cdots. \tag{3}$$

THEOREM 4. *For every $A \in B(n, n)$, the A-generated sequence is constant at the latest from its nth member on, i.e., for every $k \geqslant n$ one has $x^{(k+1)}(A) = x^{(k)}(A)$.*

Proof. We need to show that $x^{(r+1)} \geqslant x^{(r)}$ holds for every $r \geqslant n$. So as to obtain a contradiction, let us suppose that for some $r \geqslant n$ and $i \in N$ we have $x_i^{(r)} > x_i^{(r+1)}$. This means that there exists a path p in $P_i^{(r)}$ such that its capacity is greater than the capacity of each path in $P_i^{(r+1)}$.

Hence $c(p) > x_i^{(r+1)}$. But, since $l(p) = r \geqslant n$, some nodes occur at least twice on p. This means that p contains a cycle, say C, of length $m \geqslant 1$. Now we can construct another path p', beginning at i, if we insert the cycle C twice into p. But $c(p') = c(p)$ (since $\otimes$ is idempotent) and $l(p') = r + m \geqslant r + 1$. Considering that $x_i^{(r+m)}$ is the greatest capacity of a path in $P_i^{(r+m)}$, we obtain

$$x_i^{(r+m)} \geqslant c(p') = c(p) > x_i^{(r+1)}.$$

However, this is a contradiction, since $x_i^{(r+1)} \geqslant x_i^{(r+m)}$. Thus we have

$$x^{(n)} \leqslant x^{(n+1)} \leqslant \cdots,$$

which together with (3) gives $x^{(n)} = x^{(n+1)} = \cdots$. ∎

Since we have proved that $x^{(n)}(A) = x^{(n+1)}(A)$, and $x^{(n+1)}(A)$ is by its definition equal to $A \otimes x^{(n)}(A)$, an easy consequence is:

COROLLARY. *For every matrix $A \in B(n, n)$, $x^{(n)}(A)$ is its eigenvector.*

EXAMPLE 4. We compute vectors $x^{(n)}$ for matrices from the previous examples. For

$$A = \begin{pmatrix} 3 & 1 & 8 \\ 2 & 5 & 7 \\ 4 & 3 & 5 \end{pmatrix}$$

we obtain successively

$$x^{(1)} = \begin{pmatrix} 8 \\ 7 \\ 5 \end{pmatrix}, \qquad x^{(2)} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}, \qquad x^{(3)} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}.$$

Similarly, for

$$A = \begin{pmatrix} 1 & 3 & 8 & 6 \\ 0 & 4 & 5 & 2 \\ 5 & 7 & 6 & 1 \\ 9 & 4 & 3 & 2 \end{pmatrix}$$

we have

$$x^{(1)} = \begin{pmatrix} 8 \\ 5 \\ 7 \\ 9 \end{pmatrix}, \qquad x^{(2)} = \begin{pmatrix} 7 \\ 5 \\ 6 \\ 8 \end{pmatrix}, \qquad x^{(3)} = \begin{pmatrix} 6 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \qquad x^{(4)} = \begin{pmatrix} 6 \\ 5 \\ 6 \\ 6 \end{pmatrix}.$$

THEOREM 5. *Let $A \in B(n, n)$. Then $x^{(n)}(A) = x^*(A)$.*

Proof. $x_i^*(A)$, according to its definition, is the greatest h such that a cycle can be reached from i in the graph $G(A, h)$. This means that there exists a path $p = (i, j_1, \ldots, j_k, j_{k+1}, \ldots, j_k)$ in $G(A)$, beginning at i, with capacity equal to h. Since p contains a cycle C, we can insert C several times into p and obtain in this way a path p' with length $l(p') \geqslant n$ and the same capacity. This implies that

$$x_i^{(n)} = x_i^{(l(p'))} \geqslant c(p') = h = x_i^*(A).$$

To prove the converse inequality, suppose that there is a path p of length n beginning at i, with capacity equal to $x_i^{(n)}$. But p must contain a cycle, say C; hence a cycle can be reached from the node i. Since the weight of each arc on p is at least $x_i^{(n)}$, a cycle can be reached from i in $G(A, x_i^{(n)})$. This means that $x_i^*(A) \geqslant x_i^{(n)}(A)$ and we are done. ■

Let us finish with some remarks about the computational complexity of the above method for computing the upper basic eigenvector. At the beginning, the vector $x^{(1)}(A)$ is determined by finding all row maxima, and this takes time proportional to n^2. Then the matrix A is multiplied n times by an n-vector; each multiplication is of complexity $O(n^2)$. Summarizing, the time complexity of the last method is $O(n^3)$.

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